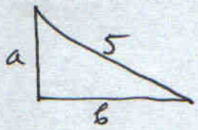
① 2010 Análisis (1)

1º// 

$A = \frac{a\cdot b}{2} = \frac{a\sqrt{25-a^2}}{2}$ $\quad F = a^2(25-a^2) = 25a^2 - a^4$

$F' = 50a - 4a^3 = 0 \quad 2a(25-2a^2) = 0 \quad \begin{cases} a = 0 \\ a = \pm\frac{5\sqrt{2}}{2} \end{cases}$

$F'' = 50 - 12a^2 \quad F''\left(\frac{5\sqrt{2}}{2}\right) < 0 \Rightarrow$ Hay máximo

$a = \frac{5\sqrt{2}}{2} \quad b = \sqrt{25 - \frac{25\cdot 2}{4}} = \sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}$

2º// $f(x) = \ln(x^2+3x)$ $\quad$ a) $f'(x_0) = 1/2$ $\quad f'(x) = \frac{2x+3}{x^2+3x} = 1/2 \Rightarrow$

$4x+6 = x^2+3x \rightarrow x^2 - x - 6 = 0 \quad x = \frac{+1 \pm \sqrt{1+24}}{2} = \begin{cases} 3 \\ -2 \end{cases}$

b) ec. tg. $P(3, \ln 18)$ $\quad y - \ln 18 = \frac{1}{2}(x-3)$

ec. normal. $\quad y - \ln 18 = -2(x-3)$

3º// $f(x) = \frac{ax^2+b}{a-x}$ para $x \neq a$ $\quad$ a) $P(2,3)$ $\quad m = -4$

$3 = \frac{4a+b}{a-2} \Rightarrow 4a+b = 3a-6 \Rightarrow a+b = -6$

$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{ax^2+b}{ax-x^2} = -a = -4 \Rightarrow \boxed{a=4 \quad b=-10}$

b) $a=2$ $b=3$ $\quad f(x) = \frac{2x^2+3}{-x+2}$ $\quad P(1,5)$ $\quad f'(x) = \frac{4x(-x+2)+2x^2+3}{(-x+2)^2} =$

$f'(x) = \frac{-2x^2+8x+3}{(-x+2)^2} \Rightarrow f'(1) = \frac{-2+8+3}{1} = 9$

$\boxed{y-5 = 9(x-1)}$

4º// Calcula $\lim_{x\to 0} \frac{e^x - e^{\operatorname{sen}x}}{x^2} = \frac{0}{0} \Rightarrow$ L'hopital. $\lim_{x\to 0} \frac{e^x - e^{\operatorname{sen}x}\cdot\cos x}{2x} = \frac{0}{0} \Rightarrow$

$\Rightarrow$ L'hopital $= \lim_{x\to 0} \frac{e^x - e^{\operatorname{sen}x}\cdot\cos^2 x + \operatorname{sen}x\, e^{\operatorname{sen}x}}{2} = \frac{1 - 1\cdot 1 + 0}{2} = 0$

5º// $f(x) = \begin{cases} e^x(x^2+ax) & \text{si } x \leq 0 \\ \frac{bx^2+c}{x+1} & \text{si } x > 0 \end{cases}$ $\Rightarrow$ a) f. es continua. en $x=0$ $\quad \lim_{x\to 0} e^x(x^2+ax) = 0 =$

$= \lim_{x\to 0} \frac{bx^2+c}{x+1} = c \Rightarrow \boxed{c=0}$

b) f. es derivable en $x=0$

$f'(x) = \begin{cases} e^x(x^2+ax) + (2x+a)e^x & x<0 \\ \frac{2bx(x+1) - bx^2 - c}{(x+1)^2} & x>0 \end{cases}$ $\quad \lim_{x\to 0^-} f'(x) = a \quad \lim_{x\to 0^+} f'(x) = -c \Rightarrow a = -c \quad \boxed{a=0}$

c) $f(1) = 3 \Rightarrow \frac{4b-b}{4} = 3 \Rightarrow 3b = 12 \quad \boxed{b=4}$

6º// $f(x) = (x+1)\sqrt[3]{3-x}$ $\quad P(-5,-8)$ $\quad P(2,3)$ $\quad f'(x) = \sqrt[3]{3-x} + \frac{1}{3}(3-x)^{-2/3}(-1)(x+1)$

$f'(-5) = 2 + \frac{4}{3}(-1)(-4) = 2 + \frac{16}{3} = \frac{22}{3}$ $\quad f'(2) = 1 + \frac{1}{3}(-1)\cdot 3 = 0$

ec. tg. $y+8 = \frac{22}{3}(x+5)$ $\quad$ ec. normal. $y+8 = -\frac{3}{22}(x+5)$

ec tg. $y-3 = 0 \Rightarrow y = 3$ (recta horiz) $\Rightarrow$ normal. $x = 2$

7º// 2010

$V = \frac{\pi r^2 h}{3}$

$V = \frac{\pi r^2 \sqrt{90^2 - r^2}}{3}$ $\quad F = \pi^2 r^4 (90^2 - r^2)$

$F' = 4\pi^2 r^3 (90^2 - r^2) - 2r(\pi r^4) = 0 \Rightarrow 90^2 \cdot 4 \cdot \pi^2 r^3 - 4\pi^2 r^5 - 2\pi^2 r^5 = 0$

$F' = 90^2 \cdot 4\pi^2 r^3 - 6\pi^2 r^5 = 0 \quad \pi^2 r^3 (90^2 \cdot 4 - 6r^2) = 0 \begin{cases} r = 0 \\ 6r^2 = 90^2 \cdot 4 \Rightarrow \\ r = \pm 30\sqrt{6} \end{cases}$

$F'' = 12 \cdot 90^2 \cdot \pi^2 r^2 - 30\pi^2 r^4 < 0.$

$r = 30\sqrt{6} \quad h = \sqrt{90^2 - 30^2 \cdot 6} = \sqrt{2700} = 30\sqrt{3}$

8º// $f(x) = \frac{x^3}{x^2 - 1} \quad x \neq \pm 1 \quad AV = \boxed{x = -1 \quad x = 1}$

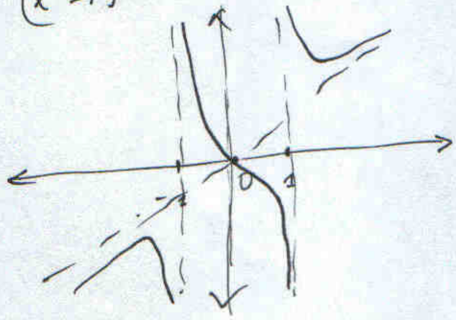
A.O. $y = mx + n \quad m = 1$

$n = \lim_{x \to \infty} \frac{x^3}{x^2 - 1} - x = \lim_{x \to \infty} \frac{x^3 - x^3 + x}{x^2 - 1} = 0 \Longrightarrow \boxed{y = x}$

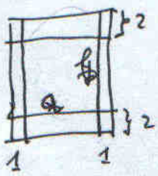
Crec. Dec. $f'(x) = \frac{3x^2(x^2-1) - 2x \cdot x^3}{(x^2-1)^2} = 0 \Longrightarrow \frac{x^4 - 3x^2}{(x^2-1)^2} = 0 \quad x^2(x^2 - 3) = 0 \begin{cases} x = 0 \\ x = \sqrt{3} \\ x = -\sqrt{3} \end{cases}$

$y' > 0 \;,\; y' < 0 \;,\; y' < 0 \;,\; y' < 0 \;,\; y' < 0 \;,\; y' > 0$

$-\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3}$



9º// $18 cm^2$



Medidas del papel. $\boxed{10 \times 5}$

$A = 18 = a \cdot b \quad b = \frac{18}{a} \quad F = (a+2)\cdot(b+4) = (a+2)\left(\frac{18}{a} + 4\right)$

$F'(a) = \left(\frac{18}{a} + 4\right) + (a+2)\left(-\frac{18}{a^2}\right) = 0 \Longrightarrow \frac{18a + 4a^2 - 18a - 36}{a^2} = 0 \Longrightarrow 4a^2 = 36 \quad a = \pm 3$

$b = 6 \quad F''(a) = \frac{8a \cdot a^2 - 2a(4a^2 - 36)}{a^4} = \frac{8a^3 - 8a^3 + 72a}{a^4} = \frac{72a}{a^4}$

$F''(3) > 0 \Longrightarrow$ Mínimo.

10º// $f: [0,4] \longrightarrow \mathbb{R} \quad f(x) = \begin{cases} x^2 + ax + b & 0 \le x \le 2 \\ cx & 2 < x < 4 \end{cases}$ $\quad f(0) = f(4) \quad f'(x) = \begin{cases} 2x + a & 0 < x < 2 \\ c & 2 < x < 4 \end{cases}$

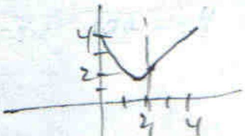
a) f(x) continua en 2. $\lim_{x \to 2^-} f(x) = 4 + 2a + b = \lim_{x \to 2^+} f(x) = 2c \Longrightarrow 4 + 2a + b = 2c$

b) f(x) derivable en 2. $\lim_{x \to 2^-} f'(x) = 4 + a = \lim_{x \to 2^+} f'(x) = c \Longrightarrow 4 + a = c$

c) $f(0) = f(4) \Longrightarrow b = 4c \Longrightarrow b = 4c.$

Luego $4 + 2a + 4c = 8 + 2a$

$\boxed{a = -3 \;,\; b = 4 \quad c = 1}$



d) extremos absolutos.

Mínimo relativo $(3/2, 7/4)$

Máximos absolutos $(0,4)$ y $(4,4)$

Mínimo absoluto $(3/2, 7/4)$ (Th de Weierstrass).

11º// $f: \mathbb{R} \longrightarrow \mathbb{R} \quad f(x) = a \operatorname{sen} x + bx^2 + cx + d \longrightarrow f'(0) = 0$ Pasa por el $(0,4)$

$f''(x) = 3 \operatorname{sen} x - 10$

$f'(x) = a\cos x + 2bx + c \Rightarrow f'(0) = 0 \Rightarrow c = 0$

$f(0) = 4 \Rightarrow d = 4 \quad f''(x) = -a \operatorname{sen} x + 2b \Longrightarrow a = -3 \quad b = -5$

12º// $f(x) = \begin{cases} e^{-x} & \text{si } x \le 0 \\ 1 - x^2 & 0 < x < 1 \\ \frac{2}{x+1} & x \ge 1 \end{cases}$

Continua y derivable en cada trozo

en 0 es continua y no derivable

en 1 no es continua ni derivable.

$f'(x) = \begin{cases} -e^{-x} & x < 0 \\ -2x & 0 < x < 1 \\ \frac{-2}{(x+1)^2} & x \ge 1 \end{cases}$

③ Matemáticas II Análisis 2010 (2ª parte)

1º// $f:(-2,\infty)\longrightarrow\mathbb{R}$ $\quad f(x)=\ln(x+2)$ $\quad F(x)=\int \ln(x+2)\,dx = \left[\begin{matrix} u=\ln(x+2) & du=\frac{1}{x+2}dx \\ dv=dx & v=x\end{matrix}\right]$

$= x\ln(x+2) - \int \frac{x}{x+2} = \left[\begin{matrix} \cancel{x} & \lfloor x+2 \\ -\cancel{x}-2 & 1 \\ -2 & \end{matrix}\right] = x\ln(x+2) - \int\left(1-\frac{2}{x+2}\right)dx =$

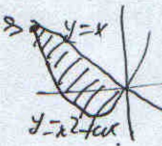
$= x\ln(x+2) - x + 2\ln(x+2) + K \qquad F(0) = 2\ln 2 + K = 0 \Rightarrow K = -2\ln 2$

$\Longrightarrow \underline{F(x) = x\ln(x+2) - x + 2\ln(x+2) - 2\ln 2}$

2º// $A=36$ entre $y=x^2+ax$ y la recta $y+x=0$ $\left\{\begin{matrix}\text{Resolvemos los puntos de corte} \\ -x = x^2+ax \Rightarrow \begin{cases} x=0 \\ x=-1-a \end{cases}\end{matrix}\right.$

El vértice de la parábola es $\left(-\frac{a}{2}, -\frac{a^2}{4}\right)$ (mínimo) luego la gráfica es

y los puntos de corte de la parábola con el eje X son: $\begin{cases} x=0 \\ x=-a \end{cases}$



$\Rightarrow 36 = \int_{-1-a}^{0} (-x-x^2-ax)\,dx = \left(-\frac{x^2}{2}-\frac{x^3}{3}-\frac{ax^2}{2}\right)_{-1-a}^{0} = -\left(-\frac{(1-a)^2}{2} - \frac{(-1-a)^3}{3} - \frac{a(-1-a)^2}{2}\right) =$

$= -\left(\frac{(-1-a)^2(-1-a)}{2} - \frac{(-1-a)^3}{3}\right) = -\frac{(-1-a)^3}{6} = 36 \Rightarrow (-1-a)^3 = (-6)^3 \Rightarrow -1-a=-6 \Rightarrow \underline{a=5}$

3º// $\int_0^{\pi^2} \operatorname{sen}(\sqrt{x})\,dx \left[\begin{matrix}\sqrt{x}=t \\ dt=\frac{dx}{2\sqrt{x}}\end{matrix}\right] = \int_0^{\pi} 2t\operatorname{sen}t\,dt = \left[\begin{matrix} u=t & du=dt \\ dv=\operatorname{sen}t\,dt & v=-\cos t\end{matrix}\right] =$

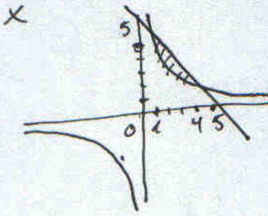
$2\left[-t\cos t + \int \cos t\,dt\right] = 2\left(-t\cos t + \operatorname{sen} t\right)_0^{\pi^2}$ deshacemos el cambio $= 2\left(-\sqrt{x}\cos\sqrt{x} + \operatorname{sen}\sqrt{x} - 0\right) =$

$= -2\pi\underset{1}{\cos\pi} + \underset{0}{\operatorname{sen}\pi} = \boxed{2\pi}$

4º// $f(x) \overset{①}{=} 5-x \quad g(x) \overset{②}{=} \frac{4}{x}$ para $x\neq 0$ $\qquad 5-x=\frac{4}{x} \Rightarrow 5x-x^2=4 \Rightarrow x^2-5x+4=0$

$x = \frac{5\pm\sqrt{25-16}}{2} = \frac{5\pm 3}{2} = \begin{cases} 4 \\ 1 \end{cases}$

x	y
0	5
5	0



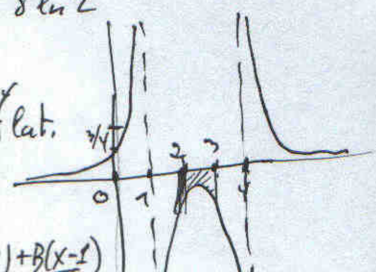
Hay que ver cuál está por encima.

$A = \int_1^4 \left(5-x-\frac{4}{x}\right)dx = \left[5x - \frac{x^2}{2} - 4\ln|x|\right]_1^4 =$

$= 20 - 8 - 4\ln 4 - 5 + \frac{1}{2} + 4\ln 1 = \boxed{7'5 - 4\ln 4} = 7'5 - 8\ln 2$

5º// $f(x) = \frac{3}{x^2-5x+4} \quad \begin{matrix} x\neq 1 \\ x\neq 4\end{matrix}$ Puntos de corte con el eje OX no hay

$f(x) = \frac{3}{(x-1)(x-4)}$ $\left\{\begin{matrix} \text{A.V } x=1 \\ \quad\;\; x=4 \\ \text{A.H } y=0\end{matrix}\right.$ límites lat.

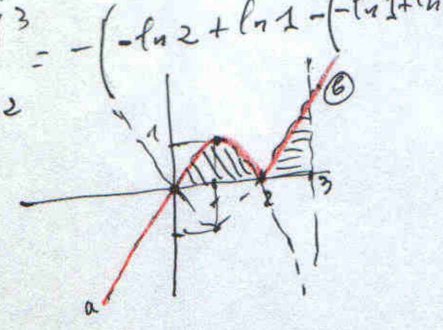


$A = -\int_2^3 \frac{3}{x^2-5x+4}\,dx = \left[\frac{3}{x^2-5x+4} = \frac{3}{(x-1)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-4)} = \frac{A(x-4)+B(x-1)}{(x-1)(x-4)}\right.$

$\left.\begin{matrix}\text{Si } x=4 \Rightarrow 3B=3 \Rightarrow B=1 \\ \text{Si } x=1 \Rightarrow -3A=3 \Rightarrow A=-1\end{matrix}\right] = -\left[-\ln|x-1| + \ln|x-4|\right]_2^3 = -\left(-\ln 2 + \ln 1 - (-\ln 1 + \ln 2)\right) =$

$= -(-2\ln 2) = \underline{2\ln 2}$

6º// $f(x) = x|2-x| = \begin{cases} -x^2+2x & x<2 \;\; ⓐ \\ x^2-2x & x\geq 2 \;\; ⓑ \end{cases}$

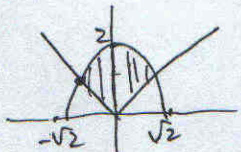


$A = \int_0^2 (-x^2+2x)\,dx + \int_2^3 (x^2-2x)\,dx = \left[-\frac{x^3}{3}+\frac{2x^2}{2}\right]_0^2 + \left[\frac{x^3}{3}-\frac{2x^2}{2}\right]_2^3 =$

$-\frac{8}{3} + 4 - 0 + \frac{27}{3} - 9 - \frac{8}{3} + 4 = \frac{11}{3} - 1 = \frac{8}{3}$

7º// — 2010 —

$f(x) = 2 - x^2 \qquad g(x) = |x|$



$A = \int_{-1}^{0} (2 - x^2 + x)\,dx + \int_0^1 (2 - x^2 - x)\,dx$

$2 - x^2 = -x$

$x^2 - x - 2 = 0 \qquad x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{+1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

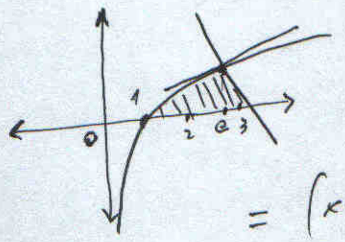
$2 - x^2 = x \qquad x^2 + x - 2 = 0 \qquad x = \frac{-1 \pm \sqrt{1+8}}{2} = \begin{cases} -2 \\ 1 \end{cases}$

$A = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} + \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = +2 - \frac{1}{3} - \frac{1}{2} + 2 - \frac{1}{3} - \frac{1}{2} = 4 - \frac{2}{3} - \frac{2}{2} = 3 - \frac{2}{3} = \frac{7}{3}$

8º// $f(x) = \ln x \longrightarrow y = -ex + 1 + e^2$ es $\perp$ en $x = e$

a) $y' = \frac{1}{x} \quad y'(e) = \frac{1}{e}$ como la pendiente de la recta normal es $-e$, es cierto.

b)



$A = \int_1^e \ln x\,dx + \int_e^{\frac{1+e^2}{e}} (-ex + 1 + e^2)\,dx =$

Punto de corte de la normal con OX $\quad 0 = -ex + 1 + e^2 \Rightarrow x = \frac{1+e^2}{e}$

$= (x\ln x - x)_1^e + \cancel{\left(-e\frac{x^2}{2} + x + e^2x\right)_e^{\frac{1+e^2}{e}}} =$ la 2ª integral se ve complicada en los cálculos

$= e\ln e - e - 1\ln 1 + 1 + \text{Área}\ \triangle = 1 + \frac{1}{e}\cdot\frac{1}{2} = 1 + \frac{1}{2e} = \frac{2e+1}{2e}$

9º// $I = \int \frac{5}{1 + \sqrt{e^{-x}}}\,dx = \left[\begin{array}{l} t^2 = e^{-x} \Rightarrow t = \sqrt{e^{-x}} \\ 2t\,dt = -e^{-x}dx \Rightarrow dx = \frac{2t\,dt}{-e^{-x}} \end{array}\right] = \int \frac{5\frac{2t\,dt}{-t^2}}{1+t} = -5\int \frac{2t\,dt}{t^2(1+t)} =$

$= -10\int \frac{t\,dt}{t^2(1+t)} = \left[\frac{t}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t} = \frac{At(1+t) + B(1+t) + Ct^2}{t^2(1+t)}\right] =$

$= \left[\begin{array}{lll} t = -1 & \Rightarrow & C = -1 \\ t = 0 & \Rightarrow & B = 0 \\ t = 1 & \Rightarrow & 2A + 2B + C = 1 \Rightarrow A = 1 \end{array}\right] = -10\left[\ln t - \ln|1+t|\right] =$

$= -10\left(\ln\sqrt{e^{-x}} - \ln|1 + \sqrt{e^{-x}}|\right) + K$

10º// $f(x) = x^2 + 4 \qquad (1, 5) \qquad f'(x) = 2x \qquad f'(1) = 2 \qquad$ a) $y - 5 = 2(x - 1)$

$y = x^2 + 4 \quad \cup\ V(0, 4) \qquad x^2 + 4 = 2x + 3$

b)

$y = 2x + 3$

x	y
0	3
$-3/2$	0

$x^2 - 2x + 1 = 0 \rightarrow (x-1)^2 = 0$

$A = \int_0^1 (x^2 + 4 - 2x - 3)\,dx = \int_0^1 (+x^2 - 2x + 1)\,dx = \left[+\frac{x^3}{3} - \frac{2x^2}{2} + x\right]_0^1 =$

$+\frac{1}{3} + 1 - 1 = \frac{1}{3}$

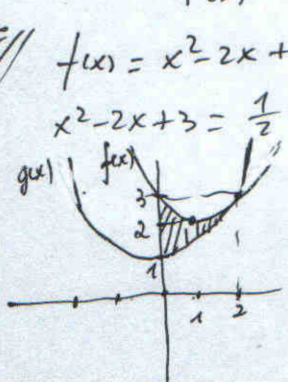
11º// $f(x) = \frac{1}{x^2 + x} \qquad \int \frac{1}{x(x+1)} = \left[\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + B(x)}{x(x+1)}\right]$

$= \left[\begin{array}{l} \text{Si } x = -1 \Rightarrow -B = 1 \\ \text{Si } x = 0 \Rightarrow A = 1 \end{array}\right] \Rightarrow \ln|x| - \ln|x+1| + K$

$F(1) = \ln 1 - \ln 2 + K = 1 \Rightarrow K = 1 + \ln 2 \Rightarrow F(x) = \ln|x| - \ln|x+1| + 1 + \ln 2$

12º// $f(x) = x^2 - 2x + 3 \qquad g(x) = \frac{1}{2}x^2 + 1 \qquad f(x) \rightarrow \cup\ V(1, 2) \qquad g(x) \rightarrow \cup\ V(0, 1)$

$x^2 - 2x + 3 = \frac{1}{2}x^2 + 1 \qquad x^2 - 4x + 4 = 0 \qquad x = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$



$A = \int_0^2 x^2 - 2x + 3 - \frac{1}{2}x^2 - 1 = \int_0^2 \frac{1}{2}x^2 - 2x + 2 = \left[\frac{1}{2}\frac{x^3}{3} - \frac{2x^2}{2} + 2x\right]_0^2 =$

$= \frac{4}{3} - 4 + 4 = \frac{4}{3}$